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Long Gravity Waves on a Viscous Fluid Flowing Down an Inclined Plane

P. SMITH

University of Keele, Staffordshire, England

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SUMMARY

The properties of permanent waves on a running stream of viscous fluid flowing down an inclined plane are investigated for small Reynolds number. The waves considered have a long wavelength compared with the depth of liquid. The approximation for the free surface in this Stokes flow problem is taken to the "third order" and extends work initiated by Mei [1]. This extension introduces additional permanent waves including cnoidal and solitary waves and also transverse strips of liquid running down the plane which appear as liquid drops in section. A qualitative analysis of the differential equation is obtained using phase-plane techniques.

1. Introduction

The steady shear flow of a viscous fluid of constant depth down an inclined plane is one of the simplest examples of steady free surface flow with the gravitational forces balancing the viscous stress. This paper presents a further study of the type and behaviour of permanent waves moving down the free surface. We shall restrict the analysis to two-dimensional disturbances.

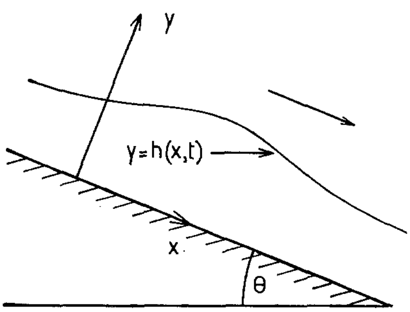

Figure 1. The coordinate scheme.

Let the plane be inclined at an angle θ to the horizontal with x- and y-axes as shown in figure 1.

The Navier–Stokes equation for the flow can be expressed as

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \operatorname{grad} \boldsymbol{v} = -\operatorname{grad} p + \rho g \boldsymbol{i} \sin \theta - \rho g \boldsymbol{j} \cos \theta + \rho \nu \nabla^2 \boldsymbol{v} , \tag{1}$$

where $\boldsymbol{v} = (u, v)$ is the velocity, p the pressure, t the time, ρ the density, ν the kinematic viscosity and g the gravitational constant. In this equation we neglect the inertia terms on the left-hand side compared with those on the right which include the pressure gradient, the gravitational force and the viscous terms. It is further assumed that disturbances take the form of long waves and, in consequence, that we can neglect x-derivatives compared with y-derivatives in the viscous terms for u and neglect entirely the viscous terms for v. In component form the momentum equations become

$$p_x = \rho g \alpha + \rho \nu u_{yy} , \tag{2}$$

$$p_y = -\rho g \beta , \tag{3}$$

where $\alpha = \sin\theta$ and $\beta = \cos\theta$. In addition u and v must satisfy the continuity equation

$$u_x + v_y = 0 \,. \tag{4}$$

Equations (2)–(4) contain the same approximation as that used by the author [2] for steady flow down an incline with an undulating bed.

The boundary conditions for continuity of velocity on the incline and for a stress-free surface $y = h(x, t)$ become

$$u = v = 0 \quad \text{on} \quad y = 0 \,, \tag{5}$$

$$\left.\begin{aligned} (p - 2\rho v u_x) h_x + \rho v (u_y + v_x) &= 0 \\ p - 2\rho v v_y + \rho v (u_y + v_x) h_x &= 0 \end{aligned}\right\} \text{ on } y = h \,, \tag{6}$$

with surface tension not included. Finally $h(x, t)$ must satisfy the kinematic surface condition

$$h_t + u h_x - v = 0 \text{ on } y = h \,. \tag{7}$$

We propose to develop an approximate solution of the system (2)–(7) which extends the work of Mei [1] to a higher order for the differential equation for the free surface elevation. Mei gives a very detailed and useful analysis of the approximation and the range of its validity in terms of the relative smallness of three parameters associated with the flow—the Reynolds, Froude and Strouhal numbers. All these parameters are small compared with unity but of comparable order to one another. Solutions are obtained by an iterative perturbation technique using expansions of the stream functions and pressure in power series in y with coefficients depending on x and t. We shall confirm that system (2)–(7) leads to the same result and also show that the next term of higher order can be readily included.

Equations (2) and (3) effectively treat the fluid as a boundary-layer with the pressure hydrostatic through the liquid. Benjamin [3] suggests that the inertial character of any disturbance to the steady flow will depend on the Reynolds number $R = P/v$, where P is the rate of volume flow per unit span of the stream. The steady flow of fluid of constant depth h_0 has a velocity distribution

$$u = gy(2h_0 - y)\alpha/(2v)$$

whence

$$P = 2h_0 u(h_0)/3 \quad \text{and} \quad R = gh_0^3\alpha/(3v^2) \,.$$

We are considering the situation in which R is small.

It should be emphasised again that the Stokes' flow problems apply only to liquids with relatively high viscosity such as certain oils or possibly to substances being processed in a molten state such as glass.

2. Formulation of the Surface Equation

From equations (2)–(5) it is easy to establish that

$$p = \rho g(q - y\beta) \,, \tag{8}$$

$$u = gy\{y(q_x - \alpha) + 2r\}/2v \,, \tag{9}$$

$$v = -gy^2\{yq_{xx} + 3r_x\}/6v \,, \tag{10}$$

where $q(x, t)$ and $r(x, t)$ are two functions to be determined by the free surface conditions (6). Thus the two linear differential equations for q and r are

$$(q - h\beta - h^2 q_{xx} - 2hr_x) h_x + hq_x - h\alpha + r - \tfrac{1}{6}h^3 q_{xxx} - \tfrac{1}{2}h^2 r_{xx} = 0 \,, \tag{11}$$

and

$$q - h\beta + h^2 q_{xx} + 2hr_x + (hq_x - h\alpha + r - \tfrac{1}{6}h^3 q_{xxx} - \tfrac{1}{2}h^2 r_{xx}) h_x = 0 \,. \tag{12}$$

It seems that these equations cannot be solved explicitly for q and r, and the following perturbation procedure is proposed to derive an approximate solution. Let

$$q=\sum_{n=1}^{\infty} q^{(n)}, \quad r=\sum_{n=1}^{\infty} r^{(n)},$$

where $q^{(n)}$ and $r^{(n)}$ are of degree n in h and its derivatives. The substitution of these expansions into (11) and (12) and the equating of terms of the same degree lead to the following iterations, to the first degree

$$-h\alpha+r^{(1)}=0, \quad q^{(1)}-h\beta=0 ;$$

to the second degree

$$q^{(1)}h_x-hh_x\beta+hq_x^{(1)}+r^{(2)}=0,$$
$$q^{(2)}+2r_x^{(1)}h-hh_x\alpha+r^{(1)}h_x=0 ;$$

to the third degree

$$q^{(2)}h_x-2hh_xr_x^{(1)}+hq_x^{(2)}+r^{(3)}-\tfrac{1}{2}r_{xx}^{(1)}h^2=0,$$
$$q^{(3)}+h^2q_{xx}^{(1)}+2r_x^{(2)}h+hh_xq_x^{(1)}+h_xr^{(2)}=0,$$

etc. These equations can be solved to give

$$q^{(1)}=h\beta, \quad q^{(2)}=-2hh_x\alpha, \quad q^{(3)}=h(hh_x+2h_x^2)\beta,$$
$$r^{(1)}=h\alpha, \quad r^{(2)}=-hh_x\beta, \quad r^{(3)}=2h(hh_x+3h_x^2)\alpha.$$

Approximations for u and v on the free surface can now be found and inserted into the kinematical surface condition. If terms up to and including the fifth degree in h and its derivatives are accounted for the surface equation becomes

$$vh_t+gh^2h_x\alpha-\tfrac{1}{3}(h^3h_x)_x\beta g+(8h^2h_x^3+8h^3h_xh_{xx}+\tfrac{2}{3}h^4h_{xxx})\alpha g=0. \tag{13}$$

If the third and fourth terms in (13) are neglected, the first order equation,

$$vh_t+gh^2h_x\alpha=0,$$

is that governing kinematic waves which has been fully studied by Lighthill and Whitham [4]. The equation with just the fourth term neglected was first obtained by Mei [1]. Takaki [5] has also derived first and second order perturbations to uniform shear flow down a vertical plate; he also includes surface tension. The case $\alpha=1$ occurs also in an exercise by Batchelor [11], p. 263. The generalisation to a non-uniform bed and to three-dimensions has been carried through by S. H. Smith [6]. The case $\alpha=0$ has also been considered by S. H. Smith [7] and similarity solutions have been obtained for certain initial-value problems involving the spread of liquid over a horizontal plane.

The full equation (13) is obviously non-linear and exhibits damping and dispersion through the second and third order derivatives of h, the latter appearing in the new term of highest degree.

3. Permanent waves

It is convenient at this point to express (13) in dimensionless form. Let h_0 represent a typical depth and x_0 a typical wavelength of the disturbance. Let $t_0=vx_0/gh_0^2$ be the representative time. The transformation

$$h=h_0H, \quad x=x_0X, \quad t=t_0T$$

changes (13) into

$$H_T+\alpha H^2H_X-\tfrac{1}{3}\beta\varepsilon(H^3H_X)_X+\alpha\varepsilon^2(8H_X^3+8HH_XH_{XX}+\tfrac{2}{3}H^2H_{XXX})H^2=0, \tag{14}$$

where $\varepsilon=h_0/x_0$.

We now look for bounded solutions of the form $H(X, T)=Q(X-CT)$ where C is the dimensionless wave speed. Thus permanent wave solutions must satisfy

$$CQ'-\alpha Q^2 Q'+\tfrac{1}{3}\beta\varepsilon(Q^3 Q')'-\alpha\varepsilon^2(8Q'^2+8QQ'Q''+\tfrac{2}{3}Q^2Q''')Q^2=0\,.$$

This equation can be integrated once to give

$$3CQ-\alpha Q^3+\beta\varepsilon Q^3 Q'-2\alpha\varepsilon^2 Q^3(4Q'^2+QQ'')=A\,, \tag{15}$$

where A is a constant.

The constant A can be interpreted as follows. The actual wave speed is Cgh_0^2/ν and equation (15) can be obtained directly by noting that the relative rate of volume flow per unit span must be constant, that is

$$\int_0^h\left(u-\frac{Cgh_0^2}{\nu}\right)dy=-\frac{Cgh_0^3 A}{3\nu} \tag{16}$$

which, when evaluated to the same approximation, supplies equation (15) again. We can assign a value to A by assuming that u is given by the shear flow of depth h_0 moving down an inclined plane, that is

$$u=gy\alpha(2h_0-y)/2\nu\,.$$

Equation (16) then gives $A=3C-\alpha$.

With the exception of one important case which will be discussed later, it seems that equation (15) cannot be integrated again in terms of standard functions. A qualitative picture of the main features of the solutions can be obtained by taking a phase plane view of (15). The singular points ($Q'=Q''=0$) must satisfy the cubic equation

$$\alpha Q^3-3CQ+3C-\alpha=0\,, \tag{17}$$

which, as we might expect from the definition of A, has one solution $Q=1$. Obviously only nonnegative roots are of physical interest, and it is assumed that $0<\alpha<1$. The other two roots are given by

$$Q=\tfrac{1}{2}[-1\pm\sqrt{\{(12C-3\alpha)/\alpha\}}]\,.$$

There is just one positive root of (17) if $C<\frac{1}{3}\alpha$, and two nonnegative roots if $C\geqq\frac{1}{3}\alpha$.

The singular points can be classified by adopting the linearising procedure given by Kaplan [8], p. 146. Suppose $Q=Q_0$ is a singular point. In the vicinity of this point the phase paths are given by

$$2\alpha\varepsilon^2 Q_0^4 Q'\frac{dQ'}{dQ}=3(C-\alpha Q_0^2)(Q-Q_0)+\beta\varepsilon Q_0^3 Q'\,.$$

The classification of the singular point depends on the sign of the discriminant

$$\Delta=\varepsilon^2 Q_0^4\{\beta^2+24\alpha(C-\alpha Q_0^2)\}\,.$$

Of the major cases, the singular point is a node if $\Delta>0$ and $C<\alpha Q_0^2$, a saddle-point if $C>\alpha Q_0^2$ and a focus if $\Delta<0$ and $C<\alpha Q_0^2$.

When $Q_0=1$, the singular point is a saddle-point if $C>\alpha$, a node if $\alpha\{1-(\beta^2/24\alpha^2)\}<C<\alpha$ and a focus if $C<\alpha\{1-(\beta^2/24\alpha^2)\}$. The second root

$$Q_0=Q_1=\tfrac{1}{2}[-1+\sqrt{\{(12C-3\alpha)/\alpha\}}]$$

is positive for $\alpha>\frac{1}{3}$. Now this positive root is a saddle-point if $\frac{1}{3}\alpha<C<\alpha$, a node if $\alpha<C<\frac{1}{8}\alpha\times\{2\gamma+5+\sqrt{(12\gamma+9)}\}$ and a focus if $C>\frac{1}{8}\alpha\{2\gamma+5+\sqrt{(12\gamma+9)}\}$ where $\gamma=\beta^2/12\alpha^2$. Thus, with the exception of the critical cases the phase plane for $Q\geqq0$ and $C\geqq\frac{1}{3}\alpha$ contains one saddle-point and a node or focus. For $C<\frac{1}{3}\alpha$, the phase plane in $Q\geqq0$ contains either a node or a focus, but the singularity must be a focus if $\gamma<\frac{4}{3}$.

Some typical phase diagrams are shown in figure 2. These were sketched with the help of the special case considered in the next section and the Poincaré perturbation technique for small β given by Minorsky [9], p. 246, which imply that the spirals are unbounded and that the phase plane does not contain limit-cycles.

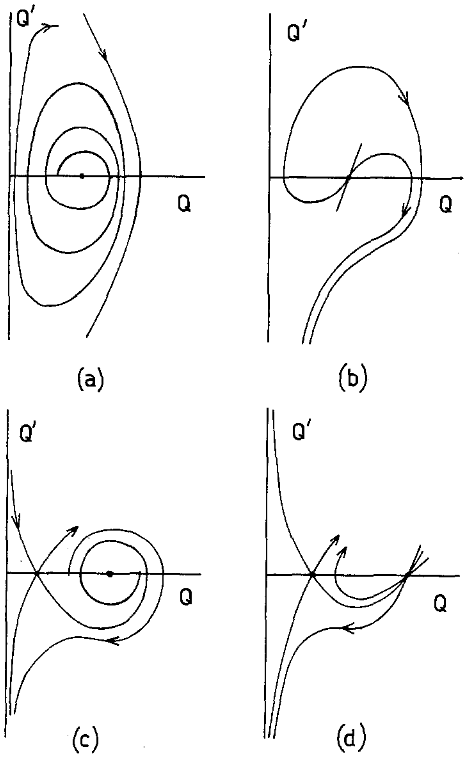

Figure 2. Phase diagrams in the four principal cases (not to scale).

The phase diagrams in figures 2(a) and 2(b) seem to be physically unrealizable; for example, in 2(a) the spiral is unbounded and so corresponds to a wave on the free surface of progressively steepening slope. In both 2(a) and 2(b), $C < \frac{1}{3}\alpha$. As Mei [1] shows the speed $\frac{1}{3}\alpha$ is that of a liquid advancing down a dry bed. Whilst there is strong circumstantial evidence for imposing the requirement $C \geqq \frac{1}{3}\alpha$, the theory is not entirely convincing on this point.

Figures 2(c) and 2(d) resemble similar diagrams given by Mei [1] for small amplitude waves, but without now the restriction to small amplitude. Some possible permanent waves are

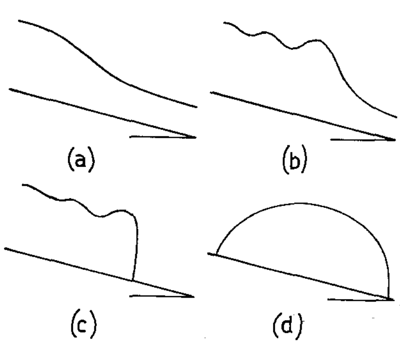

Figure 3. Some typical permanent waves.

sketched in figure 3. The inclusion of higher degree terms in equation (14) greatly extends the types of permanent waves which are possible. If the term containing ε^2 in (14) is ignored only monoclinal waves (as shown in figure 3(a)) are derived from the remainder of the equation.

4. Waves on a Liquid Flowing Down a Vertical Plane

For the vertical wall ($\alpha=1$ and $\beta=0$), the "damping" term disappears from equation (15) leaving

$$3CQ-Q^3-8\varepsilon^2Q^3Q'^2-2\varepsilon^2Q^4Q''=3C-1\,,$$

which can be integrated once to give

$$\varepsilon^2Q^8Q'^2=\tfrac{1}{2}CQ^6-\tfrac{1}{8}Q^8-\tfrac{1}{5}(3C-1)Q^5+B \tag{18}$$

where B is the integration constant. The singularities occupy the same positions as before with the saddle-point remaining a saddle-point but the focus being replaced by a centre. Equation (18) is transformed by the substitution

$$X-CT=\xi\varepsilon(8C)^{\frac{1}{2}}\,,\quad Q(X-CT)=C^{\frac{1}{2}}P(\xi)$$

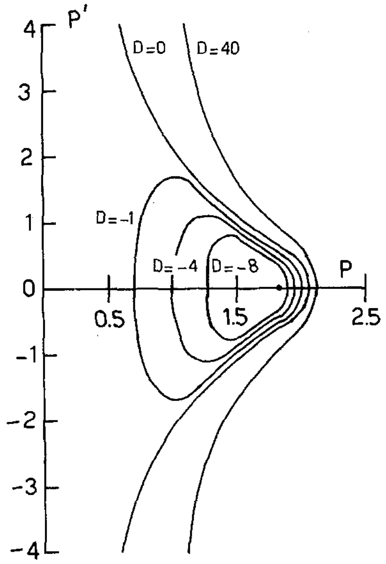

Figure 4. Phase diagram for flow down a vertical plane with one singular point.

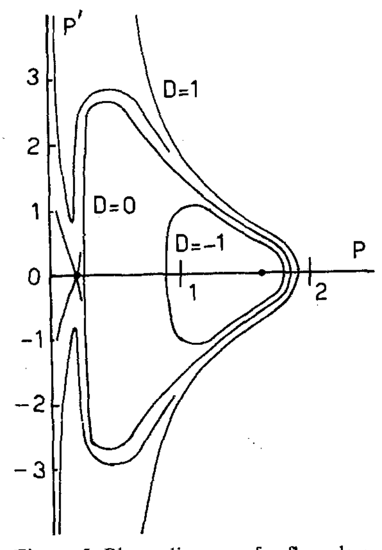

Figure 5. Phase diagrams for flow down a vertical plane with two singular points.

into the more convenient form

$$P^8 P'^2 = 4P^6 - P^8 - kP^5 + D$$

where $k = 8(3C-1)/5C^{\frac{3}{2}}$ and $D = 8B/C^4$. Phase diagrams are shown in figures 4 and 5 in the two cases $k = -1$ and $k = 1$ respectively, corresponding to one and two singular points.

In both cases closed paths represent cnoidal waves with amplitudes which can be read off the phase diagrams. The limiting case of the cnoidal wave of increasing wavelength is the solitary wave and in figure 5 it is the image of the closed separatrix which starts and ends at the saddle-point. It should be emphasised that in this context the cnoidal and solitary waves are waves on running streams. Phase paths along which $Q' \to \pm\infty$ as $Q \to 0$ correspond to a section of a liquid strip running down the vertical plane.

That the damping characteristics disappear for the vertical plane must have some implications for the stability of this situation since for small inclinations to the vertical the centre is perturbed into an unstable spiral in both instances.

Equation (18) can be integrated again if, for example, $C = \frac{1}{3}$ and $B = 0$, in which case

$$(X - CT)^2 = 8\varepsilon^2 (4C - Q^2) \tag{19}$$

ignoring the constant of integration which only gives a translation of $X - CT$. The elliptic section given by (19) is the shape of the free surface of a strip of liquid running down the plane.

Some experimental work on syrup flows down inclined planes has been recently reported by Taylor [10] and he has compared these results with the shallow flow approximation given previously by S. H. Smith [6] and P. Smith [2]. Reasonable agreement is found between the theory and experiment in his case which lends some justification to the extended approximation presented in this paper.

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